SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

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V.

INTRODUCTION OF NEW VARIABLES IN A ONE PARAMETER GROUP. LIE'S SYMBOL OF AN INFINITESIMAL TRANSFORMATION. CANONICAL FORM AND CANONICAL VARIABLES OF A ONE PARAMETER GROUP.

1. A point transformation has been defined to be an operation by which a point (x, y) is carried to the position of some point (x_1, y_1). If this operation is applied to all the points of the xy-plane simultaneously, we have a point transformation of the plane into itself. Such an operation has been represented analytically by two equations of the form

$$x_1=\varphi(x, y), \quad y_1=\psi(x, y), \tag{1}$$

where x, y represent the original points and x_1, y_1 their new positions after the transformation has been effected, and where finally the fractions φ and ψ are independent analytical functions. By introducing a parameter a into the equations (1) they will no longer represent a single transformation but an infinite number of transformations forming a family of ∞^1 transformations given by

$$x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a). \tag{2}$$

If now this family contains the inverse transformation of every transformation in it and the family possesses the property that the successive performance or product of any two transformations of the family is equivalent to a single transformation belonging to the family, the equations (2) are said to define a one parameter, finite, continuous group of transformations.

The reader will observe that the transformation (1) could have been represented by the equations

$$r_1=\rho(r, \theta), \quad \theta_1=\sigma(r, \theta), \tag{3}$$

where r, θ are the polar coördinates of the initial position of the point and r_1, θ_1 the coördinates of its final position, the functions ρ and σ being independent analytical functions. Similarly the one parameter group of transformations (2) could have been represented in polar coördinates by the two equations

$$r_1=\rho(r, \theta, \alpha), \quad \theta_1=\sigma(r, \theta, \alpha), \tag{4}$$

where α is a parameter. Hence we see that the operation (1) and the group property of the family (3) are independent of the coördinate system to which the points of the plane are referred. Accordingly, if in the equations

$$x_1=\varphi(x, y, t), \quad y_1=\psi(x, y, t) \tag{5}$$

of a one parameter group new variables $\bar{x}$, $\bar{y}$, $\bar{x}_1$, $\bar{y}_1$ are introduced, where

$$\left.\begin{aligned} \bar{x}&=\lambda(x, y), \quad \bar{y}=\mu(x, y), \\ \bar{x}_1&=\lambda(x_1, y_1), \quad \bar{y}_1=\mu(x_1, y_1), \end{aligned}\right\} \tag{6}$$

the new equations

$$x_1=\Phi(x, y, t), \quad y_1=\Psi(x, y, t) \tag{7}$$

obtained from the original ones (3) by these substitutions (6) will also represent a one parameter group.

Thus for example, it is easy to show that the ∞^1 transformations given by the equations

$$x_1=x+t, \quad y_1=\frac{xy}{x+t} \tag{8}$$

represent a one parameter group, for if (x_1, y_1) be transformed into (x_2, y_2) we have

$$x_2=x_1+t_1, \quad y_2=\frac{x_1y_1}{x_1+t_1}. \tag{9}$$

Eliminating the variables (x_1, y_1) from the equations (9) by means of the equations (8) we find

$$x_2=x+(t+t_1), \quad y_2=\frac{xy}{x+(t+t_1)}.$$

These last equations prove that the family (8) is a group, since they represent the product of two transformations of the family and are of the same form as the defining equations (8) of the family.

Now let us introduce new variables x, y into the equations of the group defined by the equations

$$\bar{x}=x/y, \quad \bar{y}=xy; \tag{10}$$

then putting

$$\bar{x}_1=x_1/y_1, \quad \bar{y}_1=x_1y_1, \tag{11}$$

and eliminating x, y, x_1, y_1 from the equations (8), (10) and (11), we have

$$\left.\begin{aligned} \bar{x}_1&=\frac{(x+t)^2}{xy}=\frac{(\sqrt{\bar{x}\bar{y}}+t)^2}{\bar{y}}, \\ \bar{y}_1&=xy=\bar{y}. \end{aligned}\right\} \tag{12}$$

The point $(\bar{x}, \bar{y})$ is carried to the position $(\bar{x}_1, \bar{y}_1)$ by the transformation (12) of the family (12), and a transformation corresponding to the parameter t_1 carries $(\bar{x}_1, \bar{y}_1)$ to the position $(\bar{x}_2, \bar{y}_2)$, say, where

$$\left.\begin{aligned}\bar{y}_2 &= \frac{(\sqrt{\bar{x}_1, \bar{y}_1}+t_1)^2}{\bar{y}_1},\\ \bar{y}_2 &= \bar{y}_1.\end{aligned}\right\} \tag{13}$$

The elimination of $\bar{x}_1$, $\bar{y}_1$ by means of the equations (12) and (13) gives,

$$\begin{aligned}\bar{x}_2 &= \frac{(\sqrt{\bar{x}\bar{y}}+t+t_1)^2}{y},\\ \bar{y}_2 &= \bar{y},\end{aligned} \tag{14}$$

the equations of the transformation of the family (12) which carries the point $(\bar{x}, \bar{y})$ directly to the position $(\bar{x}_2, \bar{y}_2)$; the form of these equations shows that this transformation belongs to the family; hence the family is a group.

2. The reader will recall that we have established in preceding sections Lie's theorem relative to the connection between the notions infinitesimal transformation and one parameter group. This relation proved to be a one-to-one correspondence of such an intimate relation that the infinitesimal transformation of a one parameter group may be taken as the complete representative of the latter without loss of generality or property. In order to make the fullest use of this equivalence of notions Lie has devised a very convenient symbol for an infinitesimal transformation. This symbol is readily constructed as follows:

Let the finite equations of the group be

$$x_1 = \varphi(x, y, t), \quad y_1 = \psi(x, y, t)\,; \tag{15}$$

and those of its infinitesimal transformation be

$$x' = x_1 + \xi(x_1, y_1)\delta t + \ldots\ldots, \quad y' = y_1 + \eta(x_1, y_1)\delta t \;\ldots\ldots; \tag{16}$$

and let the identical transformation of the group correspond to the zero value of the parameter t.

By virtue of the infinitesimal transformation (16), x_1 and y_1 receive the increments

$$\delta x_1 = \xi(x_1, y_1)\delta t, \quad \delta y_1 = \eta(x_1, y_1)\delta t. \tag{17}$$

The increment which an arbitrary function $f(x_1, y_1)$ receives by this infinitesimal transformation is found by substituting these values (17) in

$$\delta f_1 = \frac{\partial f(x_1, y_1)}{\partial x_1}\delta x_1 + \frac{\partial f(x_1, y_1)}{\partial y_1}\delta y_1,$$

to be

$$\delta f_1 = \{\xi(x_1, y_1)\frac{\partial f(x_1, y_1)}{\partial x_1} + \eta(x_1, y_1)\frac{\partial f(x_1, y_1)}{\partial y_1}\}\delta t;$$

since $f(x_1, y_1)$ becomes $f(x, y)$ when $t=0$, we have for the increment of the function $f(x, y)$

$$\delta f=\{\xi(x, y)\frac{\partial f}{\partial x}+\eta(x, y)\frac{\partial f}{\partial y}\}\delta t. \tag{18}$$

Conversely, when the increment which an arbitrary function receives by an infinitesimal transformation is known, the infinitesimal transformation itself is known ; for putting $f\equiv x$, (18) becomes

$$\delta t=\xi(x, y)\delta t\,;$$

and putting $f\equiv y$,

$$\delta t=\eta(x, y)\delta t.$$

Hence instead of defining an infinitesimal transformation by means of two equations (16) it may be characterized by the expression

$$\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y},$$

which is equal to the quotient of the increment of an arbitrary function $f(x, y)$ by the infinitesimal transformation divided by δt. For this reason Lie adopts the symbol

$$Uf\equiv\frac{\delta f}{\delta t}=\xi(x, y)\frac{\partial f}{\partial x}+\eta(x, y)\frac{\partial f}{\partial y} \tag{19}$$

as the symbol of the infinitesimal transformation (16).

If we replace $f(x, y)$ by x in this expression we find $Ux=\xi$, and if we put y for f we have $Uy=\eta$, hence the symbol may be written

$$Uf=Ux\frac{\partial t}{\partial x}+Uy\frac{\partial t}{\partial y}. \tag{20}$$

3. The question arises—How is the form of this symbol (19) affected by the introduction of new variables ? The solution of this question leads to one of the remarkable properties of the symbol.

Let the new variables be

$$\overline{x}=\lambda(x, y), \quad \overline{y}=\mu(\overline{x}, \overline{y})\,;$$

by their substitution the function $f(x, y)$ becomes $f(\overline{x}, \overline{y})$, and the principles of partial differentiation give

$$\frac{\partial t}{\partial x}=\frac{\partial f}{\partial \overline{x}}\frac{\partial \overline{x}}{\partial x}+\frac{\partial f}{\partial \overline{y}}\frac{\partial \overline{y}}{\partial x}, \qquad \frac{\partial f}{\partial y}=\frac{\partial f}{\partial \overline{x}}\frac{\partial x}{\partial y}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial y}.$$

Hence

$$Uf \equiv \xi(x,\ y)\frac{\partial f}{\partial x} + \eta(x,\ y)\frac{\partial f}{\partial y}$$

becomes

$$\xi\left(\frac{\partial t}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} + \frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial x}\right) + \eta\left(\frac{\partial f}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial y} + \frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial y}\right);$$

or

$$\left(\xi\frac{\partial \bar{x}}{\partial x} + \eta\frac{\partial \bar{x}}{\partial y}\right)\frac{\partial f}{\partial \bar{x}} + \left(\xi\frac{\partial \bar{y}}{\partial x} + \eta\frac{\partial \bar{y}}{dy}\right)\frac{\partial f}{\partial \bar{y}};$$

or finally designating Uf in the new variables by $\bar{U}f$,

$$\bar{U}f = U\bar{x}\frac{\partial f}{\partial \bar{x}} + U\bar{y}\frac{\partial f}{\partial \bar{y}}.$$

Hence if new variables, $\bar{x}$, $\bar{y}$, $\bar{x}_1$, $\bar{y}_1$ are introduced into the equations of a one parameter group,

$$x_1 = \varphi(x,\ y,\ t), \quad y_1 = \psi(x,\ y,\ t),$$

by means of the equations,

$$\bar{x} = \lambda(x,\ y), \quad \bar{y} = \mu(x,\ y),$$

$$\bar{x}_1 = \lambda(x,\ y), \quad \bar{y}_1 = \mu(x,\ y),$$

the symbol $\bar{U}f$ of the infinitesimal transformation of the new group can be determined directly by introducing the variables $\bar{x}$, $\bar{y}$ into the symbol Uf of the original group.

4. The result of the preceding paragraph gives rise to the fact that by introducing new variables $\bar{x}$, $\bar{y}$ any given infinitesimal transformation

$$Uf \equiv \xi(x,\ y)\frac{\partial f}{\partial x} + \eta(x,\ y)\frac{\partial f}{\partial y}$$

can be brought to the form of any other

$$\bar{U}f \equiv \bar{\xi}(\bar{x},\ \bar{y})\frac{\partial f}{\partial \bar{x}} + \bar{\eta}(\bar{x},\ \bar{y})\frac{\partial f}{\partial \bar{y}}.$$

In order to make this transformation it is only necessary to choose $\bar{x}$ and $\bar{y}$ as such functions of x, y that the identity

$$\bar{\xi}\frac{\partial f}{\partial \bar{x}} + \bar{\eta}\frac{\partial f}{\partial \bar{y}} = U\bar{x}\frac{\partial f}{\partial \bar{x}} + U\bar{y}\frac{\partial f}{\partial \bar{y}},$$

shall exist for all values of f. Applying the theorem of undetermined coefficients the preceding identity must break up into the two

$$U\bar{x}=\bar{\xi}, \quad U\bar{y}=\bar{\eta},$$

or written in full,

$$\xi\frac{\partial \bar{x}}{\partial x}+\eta\frac{\partial \bar{x}}{\partial y}=\bar{\xi}, \qquad \xi\frac{\partial \bar{y}}{\partial x}+\eta\frac{\partial \bar{y}}{dy}=\bar{\eta}.$$

These are two partial differential equations and we know that they have independent solutions if $\bar{\xi}$ and $\bar{\eta}$ are not both identically zero ; hence the proposed transformation is always possible since Uf is the representative of the original group and $\bar{U}f$ that of the transformed group. This proves LIE's theorem :

By introducing new variables every one parameter group of the plane can be changed into every other one parameter group of the same plane.

5. If $\bar{U}f$ has the particular form

$$\bar{U}f\equiv\frac{\partial f}{\partial \bar{y}},$$

we have the remarkable theorem that *every one parameter group of the plane can be brought to the form of a group of translations by a proper change of variables.*

These new variables $\bar{x}$ and $\bar{y}$ are found by integrating the differential equations

$$U\bar{x}\equiv\xi\frac{\partial \bar{x}}{\partial x}+\eta\frac{\partial \bar{x}}{\partial y}=0, \quad U\bar{y}\equiv\xi\frac{\partial \bar{y}}{\partial x}+\eta\frac{\partial \bar{y}}{\partial y}=1, \tag{21}$$

since in the form $\bar{U}f\equiv\dfrac{\partial f}{\partial \bar{y}}$, $\bar{\xi}=0$, and $\bar{\eta}=1$.

Now if we recall the fact that a partial differential equation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0$$

is equivalent to the ordinary differential equation

$$\frac{dx}{x}=\frac{dy}{y},$$

and that the partial differential equation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=1$$

is equivalent to the simultaneous system

$$\frac{dx}{x}=\frac{dy}{y}=df,$$

the equations (21) can be replaced by the following

$$\frac{dx}{\xi}=\frac{dy}{\eta}, \tag{22}$$

$$\frac{dx}{\xi}=\frac{dy}{\eta}=d\bar{y}. \tag{23}$$

The integration of (22) gives the function $\bar{x}$ say. This substituted in (23), the latter can be integrated by a quadrature and determines $\bar{y}$.

That the infinitesimal transformation

$$\bar{U}f\equiv\frac{\partial f}{\partial y}$$

generates a group of translations parallel to the $\bar{y}$-axis is easily shown by applying a theorem of LIE proved in a previous paragraph, namely, the theorem which shows how to construct the finite equations of a group when those of its infinitesimal transformation are known. The simultaneous system to be integrated in this case has the form

$$\frac{d\bar{x}_1}{0}=\frac{d\bar{y}_1}{1}=dt,$$

with the initial values $\bar{x}$, $\bar{y}$, 0. The integration of this simple system gives the finite equation in the form

$$\bar{x}_1=\bar{x}, \quad \bar{y}_1=\bar{y}+t;$$

that is, the group is a group of translations parallel to the $\bar{y}$-axis.

6. The theorem of the preceding paragraph can be proved by using the finite equations of the group. We have found in a previous article of this series that the finite equations

$$x_1=\varphi(x, y, t), \quad y_1=\psi(x, y, t) \tag{24}$$

of a one parameter group are found by integrating a certain simultaneous system in the form*

$$\Omega(x_1, y_1)=\Omega(x, y),$$

$$W(x_1, y_1)+t=W(x, y).$$

*The reader will observe that this is the same theorem made use of in the paragraph immediately preceding.

Now if the functions Ω and W be introduced as new variables in place of x and y, and put

$$\bar{x}=\Omega(x, y), \quad \bar{y}=W(x, y),$$

$$\bar{x}_1=\Omega(x_1, y_1), \quad \bar{y}_1=W(x_1, y_1),$$

the group takes the simple form

$$\bar{x}_1=\bar{x}, \quad \bar{y}_1=\bar{y}+t; \tag{25}$$

that is, the form of a group of translations.

The new variables $\bar{x}$ and $\bar{y}$ are called the *canonical variables* of the one parameter group (24), and (25) is called the *canonical form* of (24).

The reader will find many interesting details relative to the points and theorems here discussed, in the third chapter of LIE's lectures on differential equations.

Princeton University, 21 February, 1898.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

84. Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.

Show how to find sides, integral, fractional, and irrational for twenty-four triangles, each one containing 330 square yards.

Solution by the PROPOSER.

1. $\triangle^2=330^2=2\times2\times3\times3\times5\times5\times11\times11=(5\times11)(2\times3\times5)(2\times11)\times3$ $=55\times30.22\times3$. ∴ Sides of triangle are 25, 33, and 52.

2. $\triangle^2=330^2=(2\times5\times11)(3\times3\times11)(2\times5)=110\times99\times10\times1$. ∴ Sides of $\triangle$ are 11, 100, and 109.

3. $\triangle^2=330^2=(2\times5\times5)(3\times11)(11)(2\times3)=50\times33\times11\times6$. ∴ Sides of $\triangle$ are 17, 39, 44.

4. $330^2=(2\times3\times11)(5\times11)(2\times3)(5)=66\times55\times6\times5$. ∴ Sides of $\triangle$ are 11, 60, and 61.

5. $330^2=(2\times2\times2\times2\times3)\left(\frac{5\times11}{2}\right)(3\times5)(\frac{11}{2})=48\times27\frac{1}{2}\times15\times5\frac{1}{2}$. ∴ Sides of $\triangle$ are $20\frac{1}{2}$, 33, and $42\frac{1}{2}$.